J. DIFFERENTIAL GEOMETRY
16 (1981) 475–481

COMPACT FOUR-DIMENSIONAL EINSTEIN MANIFOLDS

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There are few known examples of compact four-dimensional Einstein manifolds (see N. Hitchin [1]), and all of them are symmetric. The purpose of this paper is to give a class of Einstein manifolds having the following properties: They are diffeomorphic to a product $S^2 \times S^2$ of two 2-spheres, not symmetric, and their sectional curvatures are not definite. The source is a theorem in [2] on a conformal diffeomorphism of a product Riemannian manifold to a 4-dimensional manifold with parallel Ricci tensor.

1. We consider a function ρ of a variable x satisfying the differential equation

$$\{\rho'(x)\}^2 = -4C\rho^3 + 2B\rho - A, \tag{1.1}$$

which is rewritten in the form

$$\{\rho'(x)\}^2 = -4C(\rho-\alpha)(\rho-\beta)(\rho-\gamma) \quad (\alpha < \beta > \gamma), \tag{1.2}$$

where A, B, C are constants, $C > 0$, and $\rho'(x)$ denotes the ordinary derivative of ρ with respect to x. Then the constants α, β and γ satisfy

$$\begin{aligned} &\alpha + \beta + \gamma = 0, \\ &2C(\alpha\beta + \beta\gamma + \gamma\alpha) = -B, \\ &4C\alpha\beta\gamma = -A, \end{aligned} \tag{1.3}$$

$\alpha > 0$, $\gamma < 0$, and β and A have the same sign.

The function ρ is a real periodic elliptic function in the range $[\beta, \alpha]$. By use of Jacobi's elliptic functions with modulus $k = \sqrt{\alpha-\beta} / \sqrt{\alpha-\gamma}$, the function ρ is expressed as

$$\rho = \frac{\beta - \gamma k^2 \operatorname{sn}^2 u}{\operatorname{dn}^2 u}, \tag{1.4}$$

where we have put $u = \sqrt{C(\alpha-\gamma)}\, x$ for simplicity. We denote by $4K$ the periodicity modulus of Jacobi's elliptic functions, and put $L = K/\sqrt{C(\alpha-\gamma)}$.

Communicated by K. Yano, February 23, 1981.

The function ρ is of period $2L$, and takes the minimum value β at $x = 0$ and the maximum value α at $x = L$. The derivative of ρ in x is given by

$$\rho'(x) = \frac{2\sqrt{C}(\alpha - \beta)(\beta - \gamma)\mathrm{sn}\, u\, \mathrm{cn}\, u}{\sqrt{\alpha - \gamma}\, \mathrm{dn}^3 u}. \tag{1.5}$$

The second derivative $\rho''(x)$ satisfies the differential equation

$$\rho''(x) = -6C\rho^2 + B, \tag{1.6}$$

and takes the values

$$\rho''(0) = 2C(\beta - \gamma)(\alpha - \gamma) > 0, \tag{1.7}$$

$$\rho''(L) = 2C(\alpha - \gamma)(\beta - \alpha) < 0 \tag{1.8}$$

in consequence of the relations (1.3).

Now let S be a 2-dimensional manifold with metric form

$$ds^2 = dx^2 + \{\rho'(x)\}^2 dy^2, \tag{1.9}$$

where y is the arc-length of a circle. We shall show that S is diffeomorphic to a 2-sphere, because ρ has the period $2L$ and $\rho'(x)$ vanishes at $x = 0$ and $x = L$. Let O and O' be the points corresponding to $x = 0$ and $x = L$ respectively.

The complementary modulus k' of k is defined by

$$k'^2 = 1 - k^2 = \frac{\beta - \gamma}{\alpha - \gamma}.$$

We define a parameter $\theta(x)$ by

$$\theta(x) = 2 \arctan\left[\mathrm{sn}\, u / (\mathrm{cn}\, u)^{k'^2}\right].$$

This parameter θ has the limits

$$\lim_{x \to 0} \theta(x) = 0, \quad \lim_{x \to L} \theta(x) = \pi,$$

and varies in the closed interval $[0, \pi]$ as x varies in $[0, L]$. Deriving θ in x, we have

$$\frac{d\theta}{dx} = \frac{2\sqrt{C(\alpha - \gamma)}\, \mathrm{dn}^3 u}{(\mathrm{cn}\, u)^{2-k^2} + (\mathrm{cn}\, u)^{k^2} \mathrm{sn}^2 u}$$

and the relation

$$\frac{d\theta}{\sin\theta} = \frac{b\,dx}{\rho'(x)},$$

where we have put $b = 2C(\alpha - \beta)(\beta - \gamma)$. The metric form of S is given by

$$ds^2 = \left(\frac{\rho'(x)}{b \sin\theta}\right)^2 \left[d\theta^2 + b^2 \sin^2\theta\, dy^2\right].$$

The expression in the brackets is the polar form of the metric of an ellipsoid of revolution. We can verify that the factor $\rho'(x)/(b\sin\theta)$ has the value

$$\left(\frac{\rho'(x)}{b\sin\theta}\right)_0 = \left(\frac{dx}{d\theta}\right)_0 = \frac{1}{2\sqrt{C(\alpha-\gamma)}},$$

and is differentiable at $x = 0$. Therefore the open subset $S - \{O'\}$ of S is conformal to the ellipsoid of revolution excluded with a point and has a differentiable structure.

On the other hand, we put

$$x' = L - x, \quad u' = K - u,$$

the former x' is the arc-length of the x-coordinate curves measured from the point O', and the latter u' is related to x' by $u' = \sqrt{C(\alpha-\gamma)}\,x'$. Since

$$\operatorname{sn}(K-u') = \frac{\operatorname{cn} u'}{\operatorname{dn} u'}, \quad \operatorname{cn}(K-u') = k'\frac{\operatorname{sn} u'}{\operatorname{dn} u'},$$
$$\operatorname{dn}(K-u') = \frac{k'}{\operatorname{dn} u'},$$

the function ρ is expressed as

$$\rho'(L-x') = \left(\beta\operatorname{dn}^2 u' - \gamma k^2 \operatorname{cn}^2 u'\right)/k'^2$$

with respect to x'. The derivative of ρ in x' is equal to

$$\rho'(L-x') = -2\sqrt{C(\alpha-\gamma)}\,(\alpha-\beta)\operatorname{sn} u' \operatorname{cn} u' \operatorname{dn} u'.$$

We define a parameter θ' by

$$\theta' = 2\arctan\left[\operatorname{sn} u'(\operatorname{dn} u')^{k^2/k'^2}/\,(\operatorname{cn} u')^{1/k'^2}\right].$$

Then we have

$$\frac{d\theta'}{dx'} = \frac{2\sqrt{C(\alpha-\gamma)}\,(\operatorname{cn} u' \operatorname{dn} u')^{k^2/k'^2}}{(\operatorname{cn} u')^{2/k'^2} + \operatorname{sn}^2 u'(\operatorname{dn} u')^{2k^2/k'^2}}$$

and the relation

$$\frac{d\theta'}{\sin\theta'} = \frac{a\,dx'}{\rho'(L-x')},$$

where we have put $a = 2C(\alpha-\beta)(\alpha-\gamma)$. The metric form of S is expressed as

$$ds^2 = \left(\frac{\rho'(L-x')}{a\sin\theta'}\right)^2\left[d\theta'^2 + a^2\sin^2\theta' dy^2\right],$$

and we can verify that the factor $\rho'(L-x')/(a\sin\theta')$ has the value

$$\left(\frac{\rho'(L-x')}{a\sin\theta'}\right)_0=\frac{1}{2\sqrt{C(\alpha-\gamma)}},$$

and is differentiable at $x'=0$. Therefore the open subset $S-\{O\}$ of S has also a differentiable structure. Hence the manifold S with metric form (1.9) is diffeomorphic to a 2-sphere S^2.

The Gaussian curvature of the manifold S is equal to

$$-\frac{\rho'''(x)}{\rho'(x)}=12C\rho. \tag{1.10}$$

2. Let $\rho_1(x)$ and $\rho_2(z)$ be elliptic functions satisfying the equations of the same type as (1.1), in which the constants B and C are common, and A may be different ones, say A_1 and A_2 for ρ_1 and ρ_2 respectively. The constants in (1.2) for ρ_1 and ρ_2 will be indicated by suffixing 1 and 2 respectively.

Let M_1 and M_2 be 2-dimensional Riemannian manifolds such as S constructed in §1 with the functions $\rho_1(x)$ and $\rho_2(z)$ for ρ respectively, and $(x^h)=(x,y)$ and $(x^p)=(z,w)$ their local coordinate systems. We consider the Pythagorean product $M=M_1\times M_2$, and denote the totality (x^h,x^p) of the coordinate systems by (x^κ). Latin indices run on the ranges

$$h,i,j,k=1,2; \qquad p,q,r,s=3,4,$$

and Greek indices run on the range from 1 to 4.

The metric tensor $g_{\mu\lambda}$, the Christoffel symbol $\{{}^{\kappa}_{\mu\lambda}\}$, the curvature tensor $K_{\nu\mu\lambda}{}^{\kappa}$ and the Ricci tensor $K_{\mu\lambda}$ of the product manifold $M=M_1\times M_2$ have pure components only. The scalar curvature κ of M is defined by

$$\kappa=\frac{1}{12}K_{\mu\lambda}g^{\mu\lambda}$$

and related to the scalar curvatures, i.e., the Gaussian curvatures κ_1 and κ_2 of M_1 and M_2 by the equation

$$6\kappa=\kappa_1+\kappa_2.$$

Taking account of (1.10) and putting

$$\sigma=\rho_1+\rho_2, \tag{2.1}$$

we see that the scalar curvature κ of M is expressed as

$$\kappa=2C\sigma.$$

The curvature tensors of the 2-dimensional manifolds M_1 and M_2 are given respectively by

$$(2.2)\qquad \begin{aligned} K_{kji}{}^{h} &= 12C\rho_1\left(\delta_k^h g_{ji} - \delta_j^h g_{ki}\right),\\ K_{srq}{}^{p} &= 12C\rho_2\left(\delta_s^p g_{rq} - \delta_r^p q_{sq}\right), \end{aligned}$$

which are the pure components of the curvature tensor $K_{\nu\mu\lambda}{}^{\kappa}$ of M.

We indicate by ∇ covariant differentiation in $M = M_1 \times M_2$. For ρ_1 in M_1 and ρ_2 in M_2, (1.1) and (1.2) are rewritten in the tensor equations

$$(2.3)\qquad \begin{aligned} |\nabla\rho_1|^2 &= -4C\rho_1^3 + 2B\rho_1 - A_1,\\ |\nabla\rho_2|^2 &= -4C\rho_2^3 + 2B\rho_2 - A_2; \end{aligned}$$

$$(2.4)\qquad \begin{aligned} \nabla_j\nabla_i\rho_1 &= \left(-6C\rho_1^2 + B\right)g_{ji},\\ \nabla_q\nabla_p\rho_2 &= \left(-6C\rho_2^2 + B\right)g_{qp}, \end{aligned}$$

where $|\nabla\rho_1|^2$ is the length of the gradient vector $\nabla_i\rho_1$. If we put $\sigma_\lambda = \nabla_\lambda\sigma$, then $\sigma_i = \nabla_i\rho_1$ and $\sigma_q = \nabla_q\rho_2$, and we have

$$(2.5)\qquad \sigma_\lambda\sigma^\lambda = |\nabla\rho_1|^2 + |\nabla\rho_2|^2.$$

For our purpose we construct a 4-dimensional Riemannian manifold M^* from the product manifold M by a conformal change of metric

$$(2.6)\qquad g^*_{\mu\lambda} = \frac{1}{\sigma^2}g_{\mu\lambda}$$

with the associated scalar field σ given by (2.1). The scalar field σ takes the minimum value $\beta_1 + \beta_2$, and we suppose that $\beta_1 + \beta_2 > 0$ or equivalently

$$A_1 + A_2 > 0$$

in order that σ be always positive.

We denote quantities of M^* by asterisking the characters corresponding to those of M. Under the conformal change (2.6), we have the transformation formulas

$$(2.7)\qquad \left\{{\kappa \atop \mu\lambda}\right\}^* = \left\{{\kappa \atop \mu\lambda}\right\} - \frac{1}{\sigma}\left(\delta_\mu^\kappa\sigma_\lambda + \delta_\lambda^\kappa\sigma_\mu - g_{\mu\lambda}\sigma^\kappa\right),$$

$$(2.8)\qquad \begin{aligned} K^*_{\nu\mu\lambda}{}^{\kappa} = K_{\nu\mu\lambda}{}^{\kappa} &+ \frac{1}{\sigma}\left(\delta_\nu^\kappa\nabla_\mu\sigma_\lambda - \delta_\mu^\kappa\nabla_\nu\sigma_\lambda + g_{\mu\lambda}\nabla_\nu\sigma^\kappa - g_{\nu\lambda}\nabla_\mu\sigma^\kappa\right)\\ &- \frac{1}{\sigma^2}\sigma_\omega\sigma^\omega\left(\delta_\nu^\kappa g_{\mu\lambda} - \delta_\mu^\kappa g_{\nu\lambda}\right). \end{aligned}$$

Referring the last equation (2.8) to the separate coordinate system (x^h, x^p), noting (2.5) and using (2.2), (2.3) and (2.4), we obtain the nontrivial components

$$
\begin{aligned}
K^*_{kjih} &= \left(A_1 + A_2 + 4C\sigma^3\right)\left(g^*_{kh}g^*_{ji} - g^*_{jh}g^*_{ki}\right),\\
K^*_{qjip} &= \left(A_1 + A_2 - 2C\sigma^3\right)g^*_{qp}g^*_{ji},\\
K^*_{srqp} &= \left(A_1 + A_2 + 4C\sigma^3\right)\left(g^*_{sp}g^*_{rq} - g^*_{rq}g^*_{sp}\right),
\end{aligned}
\tag{2.9}
$$

of the curvature tensor of M^* and the other components vanish.

The product structure $F = (F_\lambda{}^\kappa)$ of $M = M_1 \times M_2$ has eigenvalues $1, 1, -1, -1$, and composes an almost product structure together with the metric tensor $g^*_{\mu\lambda}$ of M^*, i.e.,

$$g^*_{\nu\mu}F_\lambda{}^\nu F_\kappa{}^\mu = g^*_{\lambda\kappa}.$$

We put $F^*_{\mu\lambda} = F_\mu{}^\kappa g^*_{\lambda\kappa}$, which is a symmetric tensor. Then equations (2.9) turn to the tensor equation

$$
\begin{aligned}
K^*_{\nu\mu\lambda\kappa} = {}&\left(A_1 + A_2 + C\sigma^3\right)\left(g^*_{\nu\kappa}g^*_{\mu\lambda} - g^*_{\mu\kappa}g^*_{\nu\lambda}\right)\\
&+3C\sigma^3\left(F^*_{\nu\kappa}F^*_{\mu\lambda} - F^*_{\mu\kappa}F^*_{\nu\lambda}\right).
\end{aligned}
\tag{2.10}
$$

Since $F_\lambda{}^\lambda = 0$, transvection of this equation with $g^{*\nu\kappa}$ gives

$$K^*_{\mu\lambda} = 3(A_1 + A_2)g^*_{\mu\lambda}, \tag{2.11}$$

that is, the manifold M^* is Einsteinian.

Covariantly differentiating the almost product structure $F_\lambda{}^\kappa$ with respect to the metric $g^*_{\mu\lambda}$ of M^*, substituting the formula (2.7), and taking account of the integrability $\nabla_\mu F_\lambda{}^\kappa = 0$ in M, we obtain

$$\nabla^*_\mu F^*_{\lambda\kappa} = \frac{1}{\sigma}\left(F^*_{\mu\lambda}\sigma_\kappa + F^*_{\mu\kappa}\sigma_\lambda - g^*_{\mu\lambda}F_\kappa{}^\omega\sigma_\omega - g^*_{\mu\kappa}F_\lambda{}^\omega\sigma_\omega\right). \tag{2.12}$$

The covariant derivative of the curvature tensor (2.10) of M^* is equal to

$$
\begin{aligned}
\nabla^*_\omega K^*_{\nu\mu\lambda\kappa} = 3C\sigma^2\big[&\sigma_\omega\left(g^*_{\nu\kappa}g^*_{\mu\lambda} - g^*_{\mu\kappa}g^*_{\nu\lambda}\right)\\
&+3\sigma_\omega\left(F^*_{\nu\kappa}F^*_{\mu\lambda} - F^*_{\mu\kappa}F^*_{\nu\lambda}\right)\\
&+\sigma\nabla^*_\omega\left(F^*_{\nu\kappa}F^*_{\mu\lambda} - F^*_{\mu\kappa}F^*_{\nu\lambda}\right)\big].
\end{aligned}
\tag{2.13}
$$

The covariant tensor $(F^*_{\mu\lambda})$ has components

$$(F^*_{\mu\lambda}) = \begin{pmatrix} g^*_{ji} & 0 \\ 0 & -g^*_{qp} \end{pmatrix}$$

with respect to a separate coordinate (x^h, x^p). By means of (2.12), nontrivial components of $\nabla_\mu^* F_{\lambda\kappa}^*$ are only

$$\nabla_j^* F_{ip}^* = \frac{2}{\sigma} g_{ji}^* \sigma_p, \quad \nabla_q^* F_{ip}^* = -\frac{2}{\sigma} g_{qp}^* \sigma_i. \tag{2.14}$$

The covariant derivative of the curvature tensor of M^* has for example nontrivial components

$$\nabla_\omega^* K_{kjih}^* = 12C\sigma^2\sigma_\omega\left(g_{kh}^* g_{ji}^* - g_{jh}^* g_{ki}^*\right).$$

The manifold M^* is therefore not symmetric.

Denote by $\kappa^*(X, Y)$ the sectional curvature belonging to tangent vectors X, Y. If both X and Y are tangent to one of the parts M_1 and M_2 of M as the underlying manifold of M^*, by means of the first and third expressions of (2.9), the sectional curvature $\kappa^*(X, Y)$ is equal to

$$\kappa^*(X, Y) = A_1 + A_2 + 4C\sigma^3, \tag{2.15}$$

which is always positive. On the other hand, if X and Y are tangent to M_1 and M_2 respectively, then the sectional curvature $\kappa^*(X, Y)$ is equal to

$$\kappa^*(X, Y) = A_1 + A_2 - 2C\sigma^3 \tag{2.16}$$

by means of the second of (2.9).

We suppose here $A_1 = A_2$. Then the functions $\rho_1(x)$ and $\rho_2(z)$ are the same and have the same constants, so we omit the suffices 1 and 2. The constants A, α and β are positive. By means of (1.3), the minimum of the sectional curvature (2.16) is equal to

$$\min \kappa^*(X, Y) = 2A - 16C\alpha^3 = 8C\alpha(2\alpha + \beta)(\beta - \alpha),$$

which is negative. Therefore in this case the manifold M^* has saddle points.

Bibliography

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